**Phys 410**
**Spring 2013**
**Lecture #5 Summary**
**1 February, 2013**

We discussed the motion of a charged particle in a uniform and uni-directional magnetic field $\vec{B}$, subject to the Lorentz force$\vec{F} = q\vec{v} \times \vec{B}$, where $q$ is the charge of the particle. We took $\vec{B} = B\hat{z}$ and found that Newton's second law of motion reduces to three scalar equations: $m\dot{v}_x = qv_yB$, $m\dot{v}_y = -qv_xB$, and $m\dot{v}_z = 0$. The solution for the motion along the magnetic field direction is simple: $z(t) = z_0 + v_{z0}t$. We solved the x-y plane motion using the trick of mapping this two-dimensional problem into the complex plane. Define the complex variable $\eta \equiv v_x + iv_y$, where $i = \sqrt{-1}$. The velocity of the particle is now represented as a point in the complex $\eta$ plane. The pair of coupled differential equations now reduces to a simple equation for the time evolution of $\eta$, namely $\dot{\eta} = -i\omega\eta$, and the Cyclotron frequency is defined as $\omega = qB/m$, for the charged particle of mass $m$.

The equation is solved as $\eta = \eta_0 e^{-i\omega t}$, where $\eta_0 = v_{x0} + iv_{y0} \equiv v_0 e^{i\delta}$. This equation represents uniform circular motion in the $\eta$-plane on a circle of radius $v_0$ starting at an angle $\delta$ and rotating clockwise with angular velocity $\omega$. The initial velocities are related to $v_0$ and $\delta$ as $v_{x0} = v_0 cos\delta$ and $v_{y0} = v_0 sin\delta$, and $v_0 = \sqrt{v_{x0}^2 + v_{y0}^2}$, $\delta = \tan^{-1}(v_{y0}/v_{x0})$. The resulting description of the motion can be obtained by taking the real and imaginary parts of $\eta$ as $v_x(t) = Re[\eta] = v_0\cos(\delta - \omega t)$, and $v_y(t) = Im[\eta] = v_0\sin(\delta - \omega t)$.

The trajectory of the particle in the xy-plane can be solved by a similar method. First define the complex variable $\xi \equiv x + iy$, and relate it to $\eta$ through the time derivative: $\eta = \dot{\xi}$. Integrate this equation and apply the initial conditions for $x$ and $y$ to obtain $\xi(t) = r_0 e^{i(\phi_0 - \omega t)}$, where the initial positions are written as $x_0 + iy_0 = r_0 e^{i\phi_0}$. The particle motion is described by uniform circular motion around a circle of radius $r_0$ starting at angle $\phi_0$ at angular velocity $\omega$. The resulting motion is three dimensions is helical about the magnetic field axis.

We considered several applications of these ideas to the cyclotron, the Calutron, and Whistlers in the magneto-sphere of the earth.